Journal of Pure and Applied Algebra 24 (1982) 213–216
North-Holland Publishing Company

CANCELLATION OF QUADRATIC FORMS OVER PRINCIPAL IDEAL DOMAINS

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Communicated by H. Bass
Received 16 March 1981
Revised 13 April 1981

Introduction

Let R be a commutative ring of dimension one in which 2 is invertible. It was proved in [8, Theorem 7.2] that if q is a quadratic space (i.e. a non-singular quadratic form on a finitely generated projective module) over R of Witt-index at least two, cancellation holds for q. The aim of this note is to prove a refinement of this result if R is a principal ideal domain. More precisely, we prove that if R is a principal ideal domain and q an isotropic quadratic space over R, then cancellation holds for q.[1] (We note that a quadratic form q over a Dedekind domain is isotropic if and only if its Witt-index is at least one.) We give examples to show that, in general, cancellation fails for quadratic spaces of Witt-index one over arbitrary Dedekind domains and for anisotropic spaces over principal ideal domains, thereby showing that our result is in some sense the best possible.

I am thankful to M.A. Knus for the interesting discussions I had with him during the preparation of the paper.

1. Cancellation for isotropic forms

Theorem 1. *Let R be a principal ideal domain in which 2 is invertible and let q be an isotropic quadratic space over R. If $q \perp q' \xrightarrow{\sim} q'' \perp q'$ for quadratic spaces q', q'' over R, then $q \xrightarrow{\sim} q''$.*

Proof. Let K denote the field of fractions of R. By a classical theorem of Witt, $K \otimes_R q \xrightarrow{\sim} K \otimes_R q''$ and hence there exists $\lambda \in R$, $\lambda \neq 0$ such that

$$R[1/\lambda] \otimes_R q \xrightarrow{\sim} R[1/\lambda] \otimes_R q''.$$

[1] It has been brought to my notice that Theorem 1 is contained in [9, Th. 3.1]. However, our method of proof, which is based on ideas developed in [5], and also our examples seem to be of independent interest.

We show that this implies $q \xrightarrow{\sim} q''$. By an obvious inductive argument, we may assume that $\lambda = p$ is a prime in R. We set $R[1/p] = R_p$. Let $\hat{R}$ denote the completion of R with respect to the prime ideal (p). Then $\hat{R}$ is a complete discrete valuation ring, $\hat{R}_p = \hat{R}[1/p]$ is the field of fractions of $\hat{R}$ and we have a commutative diagram

$$\begin{array}{ccc} R & \hookrightarrow & \hat{R} \\ \big\downarrow & & \big\downarrow \\ R_p & \hookrightarrow & \hat{R}_p \end{array}$$

where all the arrows are inclusions and $\hat{R} \cap R_p = R$. Since $\hat{R}$ is local, there exists, in view of [8, Theorem 8.1] an isometry

$$\psi : \hat{R} \otimes_R q \xrightarrow{\sim} \hat{R} \otimes_R q''.$$

By assumption, there exists an isometry

$$\phi : R_p \otimes_R q \xrightarrow{\sim} R_p \otimes_R q''.$$

Let $\tilde{\psi}$ and $\tilde{\phi}$ denote the extensions of ψ and ϕ respectively to isometries over $\hat{R}_p$. The element $\tilde{\psi} \cdot \tilde{\phi}^{-1}$ belongs to the orthogonal group $O_{\hat{R}_p}(q)$. Since q is isotropic, $q \xrightarrow{\sim} q_0 \perp h$ where h denotes the hyperbolic plane

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is proved in [5, proof of Proposition 3.1] that every element of the orthogonal group $O_{\hat{R}_p}(q_0 \perp h)$ is a product $\sigma_1 \cdot \sigma_2$, where $\sigma_1 \in O_{\hat{R}}(q_0 \perp h)$, $\sigma_2 \in O_{R_p}(q_0 \perp h)$, regarding $O_{\hat{R}}(q_0 \perp h)$, $O_{R_p}(q_0 \perp h)$ as subgroups of $O_{\hat{R}_p}(q_0 \perp h)$. Thus, $\tilde{\psi} \cdot \tilde{\phi}^{-1} = \sigma_1 \cdot \sigma_2$, $\sigma_1 \in O_{\hat{R}}(q_0 \perp h)$, $\sigma_2 \in O_{R_p}(q_0 \perp h)$. The isometries

$$\sigma_1^{-1} \cdot \psi : \hat{R} \otimes_R q \xrightarrow{\sim} \hat{R} \otimes_R q''$$

and

$$\sigma_2 \cdot \phi : R_p \otimes_R q \xrightarrow{\sim} R_p \otimes_R q''$$

coincide over $\hat{R}_p$ and hence define an isometry $q \xrightarrow{\sim} q''$ over R, thus completing the proof of the theorem.

Remark. Over arbitrary Dedekind domains, cancellation does not hold for isotropic quadratic spaces as is shown by the following example: Let R be a Dedekind domain such that $\operatorname{Pic} R$ contains a non-trivial element P which is a square. Since P is not free, $H(P) \not\xrightarrow{\sim} H(R)$. On the other hand $H(P \oplus R) \xrightarrow{\sim} H(R^2)$. In fact, if $P = Q \otimes_R Q$, with $Q \in \operatorname{Pic} R$, we have $P \oplus R \xrightarrow{\sim} Q \oplus Q$ and

$$H(Q \oplus Q) = ((Q_1 \oplus Q_2) \oplus (Q_1^* \oplus Q_2^*), h),$$

where $Q_1, Q_2 \xrightarrow{\sim} Q$. We have $Q_1 \oplus Q_2^* \xrightarrow{\sim} (Q_1 \otimes Q_2^*) \oplus R \xrightarrow{\sim} R^2$ is a totally isotropic direct summand of $H(Q \oplus Q)$ and hence $H(P \oplus R) \xrightarrow{\sim} H(Q \oplus Q) \xrightarrow{\sim} H(R^2)$ (see proof of Proposition 4.5 of [6]).

2. Non-cancellation for anisotropic forms

In this section, we give an example to show that cancellation does not hold in general for anisotropic forms over principal ideal domains.

Let k be any field of characteristic $\neq 2$ which admits of a quaternion division algebra H. Let P be a non-free projective ideal of $H[X, Y]$ (see [7, Proposition 1]). The norm q on the Azumaya algebra $\mathrm{End}_{H[X,Y]} P$ is a rank 4 anisotropic quadratic space over $k[X, Y]$ which is not extended from k (see [2]). However, in view of a theorem of Karoubi, q is stably extended from the reduction $\bar{q}$ of q modulo (X, Y). Thus, q and $\bar{q} \otimes_k k[X, Y]$ are stably isometric, but not isometric.

Let $k(t)$ denote the rational function field in one variable t over k and let $R = k(t)[X, Y]/(X^2 - Y^3 - t)$. The ring R is a ring of fractions of

$$S = k[t, X, Y]/(X^2 - Y^3 - t) \xrightarrow{\sim} k[X, Y]$$

and is hence a unique factorization domain [1, p. 437]. Since $\dim R = 1$, R is in fact a principal ideal domain.

Proposition 2. *Let q be the quadratic space over $k[X, Y]$ defined as above. Then $R \otimes q$ and $R \otimes \bar{q}$ are stably isometric but not isometric.*

Proof. Since q and $\bar{q}$ are stably isometric over $k[X, Y]$, $R \otimes q$ and $R \otimes \bar{q}$ are stably isometric. We shall show that they are not isometric. We recall that two Azumaya algebras of rank 4 are isomorphic if and only if their norms are isometric [3, Prop. 4.4]. Thus

$$R \otimes q \xrightarrow{\sim} R \otimes \bar{q} \Leftrightarrow \mathrm{End}_\Lambda R \otimes P \xrightarrow{\sim} \Lambda,$$

where $\Lambda = H \otimes_k R$, since $R \otimes q$ (resp. $R \otimes \bar{q}$) is the norm in $\mathrm{End}_\Lambda\ R \otimes P$ (resp. Λ). Since $\mathrm{Pic}\, R$ is trivial, this is true if and only if $R \otimes P \xrightarrow{\sim} \Lambda$ as Λ-modules. Suppose that $R \otimes P$ is free. Since R is a ring of fractions of S, there exists $g \in k[t]$ such that $S[1/g] \otimes P$ is free. Let $g = ht^n$, with $(h, t) = 1$. Let

$$\alpha : S[1/ht] \otimes P \xrightarrow{\sim} S[1/ht] \otimes H$$

be an isomorphism of $S[1/ht] \otimes H$-modules. Then

$$\mathrm{End}\, \alpha : \mathrm{End}_{S[1/ht] \otimes H} S[1/ht] \otimes P \xrightarrow{\sim} S[1/ht] \otimes H$$

is an isomorphism of $S[1/ht]$-algebras and since the reduced norm in $S[1/h] \otimes H$ is anisotropic modulo t, it follows in view of [4, Prop. 1.1] that $\mathrm{End}\, \alpha$ is defined over $S[1/h]$, i.e.

$$\mathrm{End}_{S[1/h] \otimes H} (S[1/h] \otimes P) \xrightarrow{\sim} S[1/h] \otimes H.$$

Since $\mathrm{Pic}\, S[1/h]$ is trivial, $S[1/h] \otimes P$ is free over $S[1/h] \otimes H$. Since $(h, t) = 1$, going modulo t, we get $k[X, Y]/(X^2 - Y^3) \otimes_{k[X,Y]} P$ is free over $H[X, Y]/(X^2 - Y^3)$. This is a contradiction to the fact that P is non-free over $H[X, Y]/(X^2 - Y^3)$ [6, Prop. 4.7]. This proves the proposition.

Remark. It was proved in [6, Prop. 4.7] that the non-free projective ideal P of $\mathbb{H}[X, Y]$ constructed in [7, Prop. 1] (here $\mathbb{H}$ denotes the division ring of real quaternions) remains non-free when reduced modulo $X^2 - Y^3$. Let now k be any field of characteristic $\neq 2$ which admits of a quaternion division algebra H. One could ask whether any non-free projective ideal of $H[X, Y]$ becomes free when reduced modulo a prime ideal $\mathfrak{P}$ of $K[X, Y]$ of height 1 with $K[X, Y]/\mathfrak{P}$ being regular. The above example shows that this is not in general true.

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